The decision-feedback equalizer optimization for Gaussian noise

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Abstract: The new method of decision-feedback parameters optimization for intersymbol interference equalizers is described in this paper. The error extension phenomena is well known and investigated in the decision feedback equalizers in data transmission. The existing coefficient in decision feedback depends on the receive decision risk qualification. There is proved the bit error probability can be decreased by this method for any channel with single interference sample and small Gaussian noise. The experimental results are presented for chosen type channels. The dependences of optimal feedback parameters on channel interference sample and on noise power are presented too.

Keywords: data communication, decision feedback equalizers, intersymbol interference, error analysis

1. Introduction

The intersymbol interference equalization problem with the decision-feedback use has been described in many papers. This is the decision problem concerned with incoming data receiving. It is done by taking the recognized binary data to the tapped-delay line of the transversal filter and by using these data for incoming interference erasure.

Since the data symbol is recognized, its interference with incoming next data becomes known and can be subtracted from incoming signal. The decision-feedback equalizer works properly if the decisions existing in its delay line are correct. In the case of received data error, the wrong data is taken to the feedback and calculated incoming interference differs from the real one. For binary data the existing interference is rather duplicated instead of being cancelled. In this case no cancellation or partial cancellation is proposed, by subtracting the less value than the calculated one from the signal.

The partial cancellation is proposed only when the receive decision is risky i.e. when the probability of receive error is high.

Definition: The decision is qualified to be risky (risky decision) if the signal value \(e_k\) on decision unit input is close to the decision level \(S_*, i.e. if |e_k - S_*| < \beta\).

If the probabilities of sending bit 0 and sending bit 1 are the same and the signal mean value is equal to zero, the optimal decision level \(S_* = 0\).
We assume the transmission path is composed of partially equalized channel and decision feedback equalizer as is shown on Fig. 1. The channel discrete transfer function is 
\[ Y(z) = y_0 + y_1 z^{-1} \] 
with the parameters \(|y_0| > |y_1|\).

The decision-feedback equalization with decision risk analysis is described in [3], [4] and [7]. The similar methods which do not take into equalization the interference calculated from received signal when the receive data were risky detected, have been proposed by M. Chiani [4] and by K. Hacioglu [7]. These methods do not subtract the calculated interference from existing one if risky data have been used in calculations. This reduces the error extension phenomena described in [1], [5], [7], [8]. M. Chiani [4] examined the dependence of bit error rate (BER) on risk threshold \( \beta \), searching for the best values of \( \beta \) for chosen channels to obtain the minimum BER.

If the decision \( d_k \) is risky, the algorithm presented in this paper multiplied the interference calculated from \( d_k \) by the factor \( \alpha \) and after subtracts the result from signal. I.e. if the signal value \( e_k \) on decision unit input differs from the decision level \( S^* = 0 \) less than \( \beta \), so if \( |e_k| < \beta \), the incoming interference calculated from \( d_k \) is multiplied by \( \alpha \) (where \( 0 < \alpha < 1 \)) and the result is subtracted from signal. In the case of error (\( d_k \neq a_k \)) the interference on decision unit input is multiplied by \( 1 + \alpha \) which is less than 2. The decision feedback contains an extra delay line, which remembers the risk qualifications of the decisions existing in a normal delay line of this feedback. The optimal value of risk level \( \beta \) as an analytic function of factor \( \alpha \) is given in this paper for channels with single interference sample. The graphic dependences of optimal \( \alpha \) and \( \beta \) on parameters \( y_0, y_1 \) and on noise power \( \sigma^2 \) are presented.

\[ Y(z) = y_0 + y_1 z^{-1} \]  

Figure 1. Assumed channel and equalizer

2. The risk level calculation

Let assume the noise probability density function to be even and the probability of binary data values in the transmitting sequences are equal 0.5. Let the bit zero be represented by negative pulse \( (a_k = -1) \) and the bit one be represented by positive pulse \( (a_k = 1) \).

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The first step is to find the dependence of bit error probability on parameters \( \alpha \) and \( \beta \) for the system shown on Fig. 1 with discrete transfer function given by equation

\[ Y(z) = y_0 + y_1 z^{-1}. \]  

(1)
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The sample of the channel response on input signal in time \( t=kT \) is equal \( v_k = a_k y_0 + a_{k-1} y_1 + z_k \) where \( z_k \) is the noise sample.

If the tap gain of digital delay line is \( y_1 \), the value \( e_k \) on the decision module input is [6]
\[
e_k = a_k y_0 + (a_{k-1} - d_{k-1}) y_1 + z_k
\]
(2)
when the decision \( d_{k-1} \) is not qualified to be risky, or
\[
e_k = a_k y_0 + (a_{k-1} - \alpha d_{k-1}) y_1 + z_k
\]
(3)
for risky decision \( d_{k-1} \) (0 < \( \alpha < 1 \)).

Let \( f_z(z) \) be the probability density function of the noise \( z \). We assume
- \( f_z(z) \) is the even function, i.e. \( f_z(-z) = f_z(z) \),
- the probabilities of \( a_k = 1 \) and \( a_k = -1 \) are equal.

Next we will assume the white Gaussian noise, so
\[
f_z(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}.
\]
The sign of \( e_k \) determines the decision \( d_k \). If \( e_k > 0 \) then the assumed \( d_k = 1 \), otherwise \( d_k = -1 \). If \( |e_k| < \beta \) i.e. if \( e_k \) is close to zero, the decision \( d_k \) is assumed to be risky and in the next time period the calculated interference is compensated carefully, using (3) with coefficient 0 < \( \alpha < 1 \). We will find the optimal values \( \alpha^* \) and \( \beta^* \) giving minimum probability of error for assumed white Gaussian noise power \( \sigma \) and for assumed channel factor \( y_1/y_0 \).

Therefore we will obtain the probabilities of decision \( d_k \) error (\( d_k \neq a_k \)) in the case of decision \( d_{k-1} \) was correct (\( d_{k-1} = a_{k-1} \)) or errors (\( d_{k-1} \neq a_{k-1} \)) and was assumed risky (\( |e_{k-1}| < \beta \)) or not (\( |e_{k-1}| \geq \beta \)). So we consider the equalizer feedback as a first order Markov chain with four states as is shown on Fig. 2:

- \( S_1 \), the decision \( e_{k-1} \) is correct and non risky,
- \( S_2 \), the decision \( e_{k-1} \) is false and non risky,
- \( S_3 \), the decision \( e_{k-1} \) is correct and risky,
- \( S_4 \), the decision \( e_{k-1} \) is false and risky.

Each state \( S_i \) (\( i=1, 2, 3, \) and \( 4 \)) has:
- the stationary probability \( q_i \), that the equalizer is in state \( S_i \),
- the probabilities \( p_{ij} \) (\( j=1, 2, 3, \) and \( 4 \)), that the next state will be \( S_j \),
- the error probability \( P_{ei} \).

The above described probabilities depend on \( \alpha, \beta, y_0/\sigma \) and on \( y_1/\sigma \).

![Figure 2. Four-state model of the equalizer feedback](image-url)
For further investigations we assume, the positive values $y_0$ and $y_1$. This assumption doesn’t limit the consideration generality. For negative values $y_0$ and $y_1$ the appropriate signs preceding the variables $y_0$ and $y_1$ in the next equations will change minus to plus and plus to minus, giving the same final result (9).

Let $y_0 > y_1 > 0$ and

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt.$$ 

So for $S_1$ we have:

$$p_{11} = Q\left(\frac{-y_0 + \beta}{\sigma}\right),$$

$$p_{21} = Q\left(\frac{y_0 + \beta}{\sigma}\right),$$

$$p_{31} = Q\left(\frac{y_0 - \beta}{\sigma}\right) - Q\left(\frac{y_0}{\sigma}\right),$$

$$p_{41} = Q\left(\frac{y_0}{\sigma}\right) - Q\left(\frac{y_0 + \beta}{\sigma}\right),$$

$$P_{e1} = p_{21} + p_{41} = Q\left(\frac{y_0}{\sigma}\right). \quad (4)$$

For $S_2$ we have:

$$p_{12} = \frac{1}{2} Q\left(\frac{-y_0 - 2y_1 + \beta}{\sigma}\right) + \frac{1}{2} Q\left(\frac{-y_0 + 2y_1 + \beta}{\sigma}\right),$$

$$p_{22} = \frac{1}{2} Q\left(\frac{y_0 - 2y_1 + \beta}{\sigma}\right) + \frac{1}{2} Q\left(\frac{y_0 + 2y_1 + \beta}{\sigma}\right),$$

$$p_{32} = \frac{1}{2} \left[ Q\left(\frac{y_0 - 2y_1 - \beta}{\sigma}\right) - Q\left(\frac{y_0 - 2y_1}{\sigma}\right) \right] + \frac{1}{2} \left[ Q\left(\frac{y_0 + 2y_1 - \beta}{\sigma}\right) - Q\left(\frac{y_0 + 2y_1}{\sigma}\right) \right],$$

$$p_{42} = \frac{1}{2} \left[ Q\left(\frac{y_0 - 2y_1}{\sigma}\right) - Q\left(\frac{y_0 - 2y_1 + \beta}{\sigma}\right) \right] + \frac{1}{2} \left[ Q\left(\frac{y_0 + 2y_1}{\sigma}\right) - Q\left(\frac{y_0 + 2y_1 + \beta}{\sigma}\right) \right],$$

$$P_{e2} = p_{22} + p_{42} = \frac{1}{2} \left[ Q\left(\frac{y_0 - 2y_1}{\sigma}\right) + Q\left(\frac{y_0 + 2y_1}{\sigma}\right) \right]. \quad (5)$$

For $S_3$ we have:

$$p_{13} = \frac{1}{2} Q\left(\frac{-y_0 - (1-\alpha)y_1 + \beta}{\sigma}\right) + \frac{1}{2} Q\left(\frac{-y_0 + (1-\alpha)y_1 + \beta}{\sigma}\right),$$

$$p_{23} = \frac{1}{2} Q\left(\frac{y_0 - (1-\alpha)y_1 + \beta}{\sigma}\right) + \frac{1}{2} Q\left(\frac{y_0 + (1-\alpha)y_1 + \beta}{\sigma}\right),$$
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The transfer function is given by (1). These samples are normalized for their energy equal to one, where

\[ \minimize \text{error probability} \]

Feedback can decrease the bit error probability.

A functions of channel parameters and to find how much the optimization of the decision

For \( S_4 \) we have:

\[ p_{14} = \frac{1}{2} Q\left(\frac{y_0 - (1 + \alpha) y_1 + \beta}{\sigma}\right) + \frac{1}{2} Q\left(\frac{-y_0 + (1 + \alpha) y_1 + \beta}{\sigma}\right), \]

\[ p_{24} = \frac{1}{2} Q\left(\frac{y_0 - (1 + \alpha) y_1 + \beta}{\sigma}\right) + \frac{1}{2} Q\left(\frac{y_0 + (1 + \alpha) y_1 + \beta}{\sigma}\right), \]

\[ p_{34} = \frac{1}{2} Q\left(\frac{y_0 - (1 + \alpha) y_1 - \beta}{\sigma}\right) - \frac{1}{2} Q\left(\frac{y_0 - (1 + \alpha) y_1 + \beta}{\sigma}\right) - \frac{1}{2} Q\left(\frac{y_0 + (1 + \alpha) y_1 - \beta}{\sigma}\right) - \frac{1}{2} Q\left(\frac{y_0 + (1 + \alpha) y_1 + \beta}{\sigma}\right), \]

\[ p_{44} = p_{24} + p_{34} = \frac{1}{2} Q\left(\frac{y_0 - (1 + \alpha) y_1}{\sigma}\right) + Q\left(\frac{y_0 + (1 + \alpha) y_1}{\sigma}\right). \]

The stationary probabilities \( q_i \) (i=1, 2, 3, and 4) are calculated to satisfy the equations

\[ \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}, \]

\[ q_1 + q_2 + q_3 + q_4 = 1. \]

The error probability \( P_e \) (probability of \( d_k \neq a_k \)) is given by formula

\[ P_e = P_{e1} q_1 + P_{e2} q_2 + P_{e3} q_3 + P_{e4} q_4, \]

where \( P_{e1} \) to \( P_{e4} \) are given by (4) to (7).

The problem is to find the optimal \( \alpha^*(y_1/y_0, \sigma) \) and \( \beta^*(y_1/y_0, \sigma) \), that for assumed \( y_0, y_1, \sigma \) minimize the error probability \( P_e \) given by (9).

3. Computational experiment

The aim of the experiment is to find the optimal values of \( \alpha^*(y_1/y_0, \sigma) \) and \( \beta^*(y_1/y_0, \sigma) \), as a functions of channel parameters and to find how much the optimization of the decision feedback can decrease the bit error probability.

The channels chosen in experiment were described by two pulse response samples so its transfer function is given by (1). These samples are normalized for their energy equal to one,
so \( y_0^2 + y_1^2 = 1 \). Sample \( y_1 \) is the interference sample and sample \( y_0 \) is the main sample. If there is \( |y_1| < |y_0| \), then \( |y_1| = \sqrt{1 - y_0^2} \), where \( \frac{1}{\sqrt{2}} < y_0 \leq 1 \). First the value of \( y_0 \) was increased by some constant and next the \( y_1 \) was calculated.

For such designed channels and the assumed noise power \( \sigma^2 \) the value \( \alpha \) \((0 \leq \alpha < 1)\) and the value \( \beta \) \((0 \leq \beta < y_0)\) were chosen to find the point \((\alpha^*, \beta^*)\) which minimizes the bit error probability \( P_e \) given by (4). The value \( \beta_0^* \) minimizing the bit error probability \( P_e \) for \( \alpha = 0 \) was calculated too, so the presented in this paper method can be compared with the method presented by Chiani [4].

### 4. Results and conclusion

The optimal values of \( \alpha^*(y_1/y_0, \sigma) \) and \( \beta^*(y_1/y_0, \sigma) \) as well as the dependence of \( \alpha^* \) and \( \beta^* \) on \( y_1/y_0 \) are shown on Fig. 3 and Fig. 4 for SNR = 6, 8, 10, 12, 14, and 17 dB. Optimal value of \( \alpha \) is between 0.3 and 0.4 for low interference \((y_1/y_0 < 0.2)\), between 0.2 and 0.4 for medium interference \((0.2 < y_1/y_0 < 0.6)\) and between 0.1 and 0.7 for high interference \((0.6 < y_1/y_0 < 0.95)\). If the interference is high the optimal \( \alpha \) increases from about 0.15 for high SNR to 0.45 or 0.7 (SNR≈6dB) and decreases to about 0.4 for higher noise (SNR<1dB). The optimal values of \( \beta \) decreases from about 0.75 for SNR=1dB to about 0.3 for SNR=5dB and to 0.01 for high SNR. The high error extension is in case of low SNR and high interference. For this case the optimal feedback parameters are \( \alpha^* \approx 0.5 \) and \( 0.2 < \beta^* < 0.7 \).

The probability decrease is shown on Fig. 5 and it can be up to 40% for low noise or up to 5% for SNR<7dB. The optimization of \( \alpha \) and \( \beta \) gives the best results for medium interference \((y_1/y_0 = 0.6)\) independently of SNR.

Following M. Chiani [4] we can use the optimal values \( \beta_0^* \) minimizing the bit error probability for \( \alpha = 0 \). The error probability decrease given by optimization of \( \alpha \) and \( \beta \) \((P_{opt})\) in comparison of only \( \beta \) optimization assuming \( \alpha = 0 \) \((P_1)\) can be up to 15% as is shown on Fig. 6.

![Figure 3. The optimal values of α*(y1/y0, σ) for SNR=6, 8, 10, 12, 14, and 17 dB](image-url)
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Figure 4. The optimal values of $\beta^*$ ($y_1/y_0$, $\sigma$) for SNR = 6, 8, 10 12, 14, and 17 dB

Figure 5. The error probability decrease given by optimal $\alpha^*$ and $\beta^*$ for SNR = 6, 8, 10 12, 14, and 17 dB

Figure 6. The error probability decrease given by optimal $\alpha^*$ instead of $\alpha = 0$ for SNR = 6, 8, 10 12, 14, and 17 dB
5. Summary

There was proved for channels with discrete transfer functions $Y(z) = y_0 + y_1 z^{-1}$ i.e. with one interference sample $y_1$ ($|y_1| < |y_0|$) and with white Gaussian noise. The decision feedback optimization using the risk threshold $\beta$ and erasure factor $\alpha$ optimization decreases the bit error rate. The proposed optimization of the decision feedback gives the bit error rate (BER) decreasing 1% to 40% depending on channel $(y_1/y_0)$ and on noise power. If the decision feedback equalizes more than one interference sample, the feedback parameters can be optimized separately for each sample $y_i$ taking $y_i/y_0$ into consideration for $i = 1, 2, \ldots$.

References