Central-symmetrical property analysis on circularly orthogonal moments

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Abstract: In this research, we have proposed the central-symmetrical property of image reconstructions from Zernike moments and pseudo-Zernike moments. We conducted the image reconstructions from the odd, even, and complete sets of Zernike moments and pseudo-Zernike moments, and verified the proposed central-symmetrical property. We have concluded that if the original image is centrally symmetrical, the image reconstructions from even order sets are identical to those from the corresponding complete order sets of either Zernike or pseudo-Zernike moments.

Keywords: Central-symmetrical property, circularly orthogonal moments, Zernike moments, pseudo-Zernike moments

1. Introduction

In this research, two types of circularly orthogonal moments, namely Zernike moments and pseudo-Zernike moments, are analyzed for their central-symmetrical property.

Zernike moments are composed of a basis set of complex orthogonal Zernike polynomials, while the pseudo-Zernike moments, with a similar structure as Zernike moments, are based on a set of complex orthogonal pseudo-Zernike polynomials. Due to several advanced fundamental properties, particularly the distinctive property of being invariant to rotations and reflections, both Zernike moments and pseudo-Zernike moments have been the subjects of extensive theoretical studies since they were introduced by Teague in 1980 [1], and Teh and Chin in 1988 [2], respectively. A wide range of applications have been developed for digital image analysis, processing, and recognition in recent years, such as character recognition [3], sketched symbols recognition [4], aircrafts recognition [5], shape retrieval [6], rose variation recognition [7], distinguishing the soft liver tissues [8], description of moving objects [9], face recognition [10], palmprint verification [11], image denoising [12], and watermarking [13]. For a general study of these two important orthogonal moments defined in a circular domain, we refer to [14, 15, 16].

Recently, the investigations on image reconstructions from continuous orthogonal moments have led to the conclusion that the even orders of several orthogonal moments describe most of the image characteristics, while those of the odd orders only present very limited information of the original image [17, 18]. To examine this issue further, in this paper, we have conducted a research study on the central-symmetrical property of Zernike moments and pseudo-Zernike moments. By analyzing the relationships of four quadrants of an image during its reconstruction process, we have concluded that the images represented from even orders of Zernike and pseudo-Zernike moments are centrally symmetrical. Therefore, the
more central-symmetrical an image is, the more similar a reconstructed image made from just the even orders is to a reconstructed image made from the completed order sets of Zernike moments and pseudo-Zernike moments. If an image is centrally symmetrical, the reconstructed image from even orders will be identical to the image reconstructed from the corresponding complete order sets of Zernike or pseudo-Zernike moments.

In Section 2, we will review the fundamental characteristics of Zernike and pseudo-Zernike moments. The central-symmetrical property for both Zernike and pseudo-Zernike moments is proposed in Section 3. To verify the proposed central-symmetrical property, the image reconstructions from odd orders, even orders, and the complete sets of Zernike and pseudo-Zernike moments are conducted in Section 4. Finally, we will make our conclusions and remarks in Section 5.

2. Zernike moments and pseudo-Zernike moments

2.1. Zernike moments

A Zernike polynomial, \( V_{pq}(x,y) \), is formed by a set of complex orthogonal and square integrable functions with simple rotational property, defined over a unit circle

\[
V_{pq}(x,y) = R_{pq}(\rho) \exp(jq\theta), \quad x^2 + y^2 \leq 1, (1)
\]

where \( \rho = \sqrt{x^2 + y^2} \) and \( \theta = \arctan(y/x) \).

In (1) the radial polynomial \( R_{pq}(\rho) \) can be expressed as

\[
R_{pq}(\rho) = \sum_{s=0}^{(p-|q|)/2} (-1)^s \frac{(p-s)!}{s! \left( \frac{p+|q|}{2} - s \right)! \left( \frac{p-|q|}{2} - s \right)!} \rho^{p-2s}, (2)
\]

where \( p \geq |q| \) and \( p - q = \text{even} \).

The orthogonality of the radial polynomial \( R_{pq}(\rho) \) leads to the orthogonal relation of Zernike polynomial set \( V_{pq}(x,y) \) in the two-dimensional circular domain

\[
\int \int_D V_{pq}^* (x,y) V_{p'q'} (x,y) dxdy = \frac{\pi}{p + 1} \delta_{pp'} \delta_{qq'}, (3)
\]

where \( \delta_{pp'} = 1 \) if \( p = p' \) and 0 otherwise, and \( D \) represents the circular domain.

Zernike moment of order \( p \) with repetition \( q \) is defined as

\[
A_{pq} = \int \int_D f(x,y)V_{pq}^*(x,y)dxdy, (4)
\]

where * denotes complex conjugate.

When digitalizing an analog image \( f(x,y) \) into its discrete version \( f(x_i,y_j) \), the double integrations in (4) need to be approximated by double summations, such as

\[
\hat{A}_{pq} = \sum_{(x_i,y_j) \in D} f(x_i,y_j)V_{pq}^*(x_i,y_j)\Delta x \Delta y, (5)
\]

where \( \Delta x \) and \( \Delta y \) are the sampling intervals in the \( x \) and \( y \) directions, respectively. However, the accuracy of (5) will decrease as the orders of the Zernike moments increase. A comprehensive discussion on the issue was addressed in [19], and we apply the numerical scheme adopted in [20][18]

\[
\hat{A}_{pq} = \sum_{x_i^2 + y_i^2 \leq 1} f(x_i,y_j)h_{pq}(x_i,y_j) (6)
\]
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to compute the Zernike moments in this research, where

$$h_{pq}(x_i, y_j) = \int_{x_i - \Delta x}^{x_i + \Delta x} \int_{y_j - \Delta y}^{y_j + \Delta y} V_{pq}^*(x, y) dx dy.$$  (7)

By dividing each image pixel into \(k \times k\) sub-regions with the same weight to simulate the double integrations in (7), we can increase the accuracy of Zernike moment computation significantly.

Due to the orthogonality and completeness of Zernike functions \(V_{pq}(x, y)\), we can reconstruct an image from an infinite set of digitalized Zernike moments

$$f(x, y) = \sum_{p=0}^{\infty} \sum_{q=-p}^{p} \frac{p + 1}{\pi} \hat{A}_{pq} V_{pq}(x, y)$$  (8)

where \((p + 1)/\pi\) is the normalizing constant. However, in practice, we can only reconstruct an image from a finite set of Zernike moments

$$f(x, y) = \sum_{p=0}^{T} \sum_{q=-p}^{p} \frac{p + 1}{\pi} \hat{A}_{pq} V_{pq}(x, y)$$  (9)

where \(T\) is a truncated parameter indicating the maximum order of Zernike moments utilized to reconstruct the image function \(f(x, y)\).

2.2. Pseudo-Zernike moments

A pseudo-Zernike polynomial, the modified version of a Zernike polynomial, was derived by Bhatia and Wolf [21] with a definition of

$$V_{nm}(x, y) = R_{nm}(\rho) exp(jq\theta), \quad x^2 + y^2 \leq 1,$$  (10)

where \(\rho = \sqrt{x^2 + y^2}\) and \(\theta = \arctan(y/x)\). The radial polynomial \(R_{nm}(\rho)\) of pseudo-Zernike function in (10) is defined as

$$R_{nm}(\rho) = \sum_{s=0}^{n-|m|} (-1)^s \frac{(2n + 1 - s)!}{s! (n + |m| + 1 - s)! (n - |m| - s)!} \rho^{n-s},$$  (11)

where \(n\) is a non-negative integer, \(m\) is an integer, and \(|m| \leq n\).

According to the restriction of the radial polynomial, a set of pseudo-Zernike polynomial contain \((n + 1)^2\) linearly independent polynomials when the maximum order equals \(n\), while a set of Zernike polynomials consists of \((n + 1)(n + 2)/2\) linearly independent polynomials of degree \(\leq n\). Therefore, the time consumption for calculating pseudo-Zernike moments is nearly twice as much as the time required to compute Zernike moments.

The definition of a pseudo-Zernike moment of order \(n\) with repetition \(m\) is

$$A_{nm} = \int \int_D f(x, y)V_{nm}^*(x, y) dx dy.$$  (12)

Similar to the Zernike moments, an image \(f(x, y)\) can be reconstructed from an infinite set of pseudo-Zernike moments

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{n + 1}{\pi} A_{nm} V_{nm}(x, y).$$  (13)
With a given maximum order of pseudo-Zernike moments, \( T \), we can reconstruct an approximation of image function \( f(x, y) \) from a finite set of digitalized pseudo-Zernike moments

\[
f(x, y) = \sum_{n=0}^{T} \sum_{m=-n}^{n} \frac{n + 1}{\pi} \hat{A}_{nm} V_{nm}(x, y).
\]  

(14)

3. The central-symmetrical property

In this section, we will analyze the central-symmetrical property of a reconstructed image from even orders of Zernike or pseudo-Zernike moments.

According to (9), only the Zernike polynomial \( V_{pq}(x, y) \) is related to the coordinate axes, \( x \) and \( y \), where \( (p + 1)/\pi \) and Zernike moments \( \hat{A}_{pq} \) can be considered as parameters. Referring to (1), we can rewrite the Zernike polynomials \( V_{pq}(x, y) \) with Euler’s formula as

\[
V_{pq}(x, y) = VR_{pq}(x, y) + jVI_{pq}(x, y),
\]  

(15)

where \( VR_{pq}(x, y) \) and \( VI_{pq}(x, y) \) are the real and imaginary parts of \( V_{pq}(x, y) \), respectively.

Figure 1 shows an image divided into four quadrants, where the centre of the image is taken as the origin.

\[ \begin{array}{cc}
S2 & S1 \\
S3 & S4
\end{array} \]

Figure 1. Four quadrants of a coordinate axis.

Since the values of \( R_{pq}(\rho) \) in each of the quadrants are symmetrical to either the \( x \)- or \( y \)-axis, we can calculate the values of \( V_{pq}(\rho) \) in one quadrant, then obtain the values of the other three quadrants with conjugate transformation and sign changes.

For the values of \( V_{pq}(x, y) \), the relationship between all quadrants can be divided into two situations [22]. When the repetition \( q \) is an odd number, the relationships between the first quadrant \( S1 \) and other quadrants, \( S2, S3, \) and \( S4 \) are

\[
\begin{align*}
VR_{pq}(S2) &= VR_{pq}(S1) \\
VI_{pq}(S2) &= -VI_{pq}(S1) \\
VR_{pq}(S3) &= -VR_{pq}(S1) \\
VI_{pq}(S3) &= -VI_{pq}(S1) \\
VR_{pq}(S4) &= -VR_{pq}(S1) \\
VI_{pq}(S4) &= VI_{pq}(S1)
\end{align*}
\]

When the repetition \( q \) is an even number, we have the following relationships

\[
\begin{align*}
VR_{pq}(S2) &= VR_{pq}(S1) \\
VI_{pq}(S2) &= -VI_{pq}(S1) \\
VR_{pq}(S3) &= VR_{pq}(S1) \\
VI_{pq}(S3) &= VI_{pq}(S1) \\
VR_{pq}(S4) &= VR_{pq}(S1) \\
VI_{pq}(S4) &= -VI_{pq}(S1)
\end{align*}
\]
By combining (4) and (15), we can conclude that Zernike moment $A_{pq}$ is a complex number

$$A_{pq} = AR_{pq} + jAI_{pq}. \quad (16)$$

By applying (9), we are able to reconstruct an image with complex values as well. Then, for a given $p$ and $q$, the reconstructed image can be expressed as

$$f_{pq}(x,y) = fR_{pq}(x,y) + jfI_{pq}(x,y) \quad (17)$$

where

$$fR_{pq} = \frac{p+1}{\pi} \times VR_{pq} \times AR_{pq} - \frac{p+1}{\pi} \times VI_{pq} \times AI_{pq} = fR1 - fR2, \quad (18)$$

and

$$fI_{pq} = \frac{p+1}{\pi} \times VI_{pq} \times AR_{pq} + \frac{p+1}{\pi} \times VR_{pq} \times AI_{pq} = fI1 + fI2. \quad (19)$$

To address this issue, we use a testing image shown in Figure 2, which is sized at $256 \times 256$ with 256 gray levels.

The first quadrant of Figure 3 is the same as that of Figure 2. The second quadrant of Figure 3, denoted as $Img(S2)$, is symmetrical to the first quadrant of Figure 3, denoted as $Img(S1)$, about the y-axis. We describe the relationship of $Img(S1)$ and $Img(S2)$ as

$$Img(S2) = Img(S1)[Same, Reverse],$$

where Same indicates that no change is needed in x direction, and Reverse specifies that a reverse transformation for coordinates is needed in the y direction.

Similarly, $Img(S3)$ and $Img(S4)$ can be expressed as

$$Img(S3) = Img(S1)[Reverse, Reverse]$$

$$Img(S4) = Img(S1)[Reverse, All].$$

For the reconstructed images, the value of each pixel is always a real number, so we can discard the imaginary part $fI_{pq}(x,y)$. Then, we rewrite the relationship between the first quadrant $S1$ and other three quadrants. When the repetition $q$ is an odd number, the four quadrants of a reconstructed image are:

$$f_{pq}(S1) = fR_{pq}(S1) = fR1(S1) - fR2(S1)$$

$$f_{pq}(S2) = (fR1(S1) + fR2(S1))[All, Reverse]$$

$$f_{pq}(S3) = -(fR1(S1) - fR2(S1))[Reverse, Reverse]$$

$$f_{pq}(S4) = -(fR1(S1) + fR2(S1))[Reverse, All] \quad (20)$$

Figure 2. A testing image sized at $256 \times 256$. 
When the repetition \( q \) is an even number, the four quadrants of a reconstructed image can be expressed as:

\[
\begin{align*}
    f_{pq}(S1) & = fR_{pq}(S1) = fR1(S1) - fR2(S1) \\
    f_{pq}(S2) & = (fR1(S1) + fR2(S1))[All, Reverse] \\
    f_{pq}(S3) & = (fR1(S1) - fR2(S1))[Reverse, Reverse] \\
    f_{pq}(S4) & = (fR1(S1) + fR2(S1))[Reverse, All]
\end{align*}
\]

(21)

From (21), we can conclude that an image reconstructed from even orders of Zernike moments is centrally symmetrical, and the origin is located in the center of the image.

Since pseudo-Zernike moments have a similar structure as Zernike moments, the relationship between the first quadrant of the reconstructed image and other three quadrants is the same as that of Zernike moments.

4. Image reconstructions from Zernike moments and pseudo-Zernike moments

4.1. Image reconstructions from Zernike moments

The \( k \times k \) numerical scheme, proposed in [20], is adopted to compute Zernike moments, and we will refer to [18] for the corresponding accurate verifications of Zernike and pseudo-Zernike moments.

Figure 4 shows three testing images utilized in this research. Figure 4 (a) is not a central-symmetrical image. Although Figure 4 (b) is very close to central-symmetrical visually, it is not such an image. Figure 4 (c) is a central-symmetrical image pixel by pixel. All testing images shown in Figure 4 are sized at 256 \( \times \) 256 with 256 gray levels.
To verify the central-symmetrical property of reconstructed images from partial sets of Zernike moments, we use the following equation to reconstruct images

$$\hat{f}(x, y) = \sum_{p=T1}^{T2} \sum_{q=-p}^{p+1} \frac{1}{\pi} \hat{A}_{pq} V_{pq}(x, y), \quad |q| \leq p, \quad \text{and} \quad p - q = \text{even},$$

(22)

where $T1$ and $T2$ decide the range of a partial set of Zernike moments.

The Peak Signal to Noise Ratio (PSNR) is used in our experiment as the measurement to compare the reconstructed image with its original image. PSNR is the ratio between the maximum power of the signal and the affecting noise and is image independent. The definition of PSNR is

$$\text{PSNR} = 10 \log_{10} \left( \frac{\text{MAX}_I^2}{\text{MSE}} \right),$$

(23)

where $\text{MAX}_I$ is the maximum gray level value of the image, which is 255 for our testing images. The Mean Squared Error (MSE) is defined as

$$\text{MSE} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} [f(x_i, y_j) - \hat{f}(x_i, y_j)]^2,$$

(24)

where $f(x_i, y_j)$ and $\hat{f}(x_i, y_j)$ are the original image and the reconstructed image, respectively.

Considering the restriction of order $p$ and its repetition $q$ for Zernike moments, the parity of $p$ and $q$ is consistent. For convenience, we will divide Zernike moments into even and odd order sets according to the order $p$. Figure 5 shows the structure for selecting the even order sets and odd order sets of Zernike moments. The even order sets of Zernike moments are highlighted by gold, while the odd order sets are highlighted by blue.

![Figure 4. Three testing images are sized at 256×256 with 256 gray levels.](image)

![Figure 5. The structure of using order $p$ to reconstruct an image.](image)
Figure 6. Image reconstructions from odd order sets in the first and second rows, those from even order sets in the third and fourth rows, and those from complete order sets in the last two rows.

Figure 6 illustrates some reconstructed Figure 4 (a) from odd, even, and complete sets of Zernike moments with $7 \times 7$ numerical scheme, while the corresponding PSNR values are displayed in Table 1. It can be observed that the reconstructed images from even order sets are centrally symmetrical.

We have conducted the image reconstruction performances on Figure 4 (b) with $7 \times 7$ numerical scheme as well. Figure 7 shows some of the reconstructed images from even and complete order sets. Since all images reconstructed only from odd orders of Zernike moments contain very limited image information, we have not displayed any of these images in Figure 7. The corresponding PSNR values of Figure 7 are displayed in Table 2. We can observe
that the reconstructed images from even order sets are very close to those from the complete order sets of Zernike moments either visually or measured by PSNR values.

Table 1. Corresponding PSNRs of Figure 6

<table>
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Table 2. Corresponding PSNRs of Figure 7

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<tr>
<td>T=240</td>
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</tr>
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We have also reconstructed Figure 4 (c), which is centrally symmetrical, from even and complete order sets with $7 \times 7$ numerical scheme, and all image reconstructions are illustrated in Figure 8.

MSE in (24) is applied to compare the difference of the image reconstructions from even order sets and their corresponding complete order sets. Table 3 exhibits the MSEs from different maximum order sets according to Figure 4 (a), (b), and (c), respectively, where column (a) corresponds to Figure 4 (a), column (b) corresponds to Figure 4 (b), and column (c) corresponds to Figure 4 (c). By observing the first column of Table 3, it shows that image reconstructions from even order sets are identical to those from corresponding order sets when the original image is centrally symmetrical.

By observing all results shown in Figure 6, Figure 7, Figure 8, Table 2, and Table 3, we conclude that the more central-symmetrical an image is, the closer their reconstructed
Figure 7. Image reconstructions from even order sets in the first and second rows, and those from complete order sets in the last two rows.

Images from even orders only and the completed order sets of Zernike moments. If an image is centrally symmetrical, the reconstructed image from even orders will be identical to the image reconstructed from the corresponding complete order sets of Zernike moments.

Table 3. MSEs between the reconstructed images from even and complete order sets of Zernike moments and their original images.

<table>
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<th>(b)</th>
<th>(c)</th>
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<td>0</td>
</tr>
<tr>
<td>T=220</td>
<td>2744.7</td>
<td>0.2</td>
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<td>T=240</td>
<td>2745.5</td>
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4.2. Image reconstructions from pseudo-Zernike moments

To verify the central-symmetrical property of pseudo-Zernike moments, we utilize the image reconstruction formula

\[
\hat{f}(x, y) = \sum_{p=1}^{T1} \sum_{m=-n}^{n} n + \frac{1}{\pi} A_{nm} V_{nm}(x, y), \quad |m| \leq n,
\]

where \( T1 \) and \( T2 \) indicate the upper and lower bounds of the partial sets of pseudo-Zernike moments.

Figure 9 shows the structure for selecting even order sets and odd order sets according to the repetition \( m \). The columns highlighted by gold represent the even order sets of pseudo-Zernike moments, while the blue columns represent the odd order sets.

Figure 10 shows some reconstructed Figure 4 (a) from the pseudo-Zernike moments with odd orders, even orders, and the complete orders, respectively. The corresponding PSNRs are illustrated in Table 4.

We have also reconstructed Figure 4 (b) from the even and complete order sets and shown some results in Figure 11 and the corresponding PSNR values in Table 5.

Figure 12 illustrates some reconstructed Figure 4 (c), which is centrally symmetrical, from even and complete order sets with \( 7 \times 7 \) numerical scheme.

MSE is used in Table 6, to compare the difference of the image reconstructions from even order sets and their corresponding complete order sets. Figure 4 (a), (b), and (c) are selected
Figure 9. The structure of using order $m$ to reconstruct an image

### Table 4. Corresponding PSNRs of Figure 10

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### Table 5. Corresponding PSNRs of Figure 11

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as the original images for column (a), (b), and (c) in Table 6, respectively. Referring to the first column of Table 6, it shows that image reconstructions from even order sets are identical to those from corresponding order sets when the original image is centrally symmetrical.

By observing the reconstructed images shown in Figure 10, Figure 11 and Figure 12, and Table 6, we have the same conclusion as we did for Zernike moments. If the original image
Figure 10. Image reconstructions from odd order sets of pseudo-Zernike moments in the first and second rows, those from even order sets in the third and fourth rows, and those from complete order sets in the last two rows.

is centrally symmetrical, the reconstructed images from even order sets of pseudo-Zernike moments are identical to those from the corresponding complete order sets of pseudo-Zernike moments.
Figure 11. Image reconstructions from even order sets of pseudo-Zernike moments in the first and second rows, and those from complete order sets in the last two rows.

Figure 12. Image reconstructions from even order sets in the first and second rows, and those from complete order sets in the last two rows.
Table 6. MSEs between the reconstructed images from even and complete order sets of pseudo-Zernike moments and their original images.

<table>
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<th>(c)</th>
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5. Concluding remarks

In this research, we proposed the central-symmetrical property of image reconstructions from Zernike moments and pseudo-Zernike moments. To verify validity of this property, we have performed image reconstructions with different testing images from odd, even, and complete order sets of these two circularly orthogonal moments with satisfactory results.

Our conclusion is that the more central-symmetrical an image is, the closer their reconstructed images from even orders only and the completed order sets of Zernike or pseudo-Zernike moments. If an image is centrally symmetrical, the reconstructed image from even orders will be identical to the image reconstructed from the corresponding complete order sets of either Zernike or pseudo-Zernike moments.

We expect similar central-symmetrical properties to exist on other circularly orthogonal moments.

Acknowledgment

The authors wish to thank the reviewers and Dr. Piotr Czapiewski, the Managing Editor, for their valuable comments.

References


