Is the conventional interval arithmetic correct?

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Abstract: Interval arithmetic as part of interval mathematics and Granular Computing is unusually important for development of science and engineering in connection with necessity of taking into account uncertainty and approximativeness of data occurring in almost all calculations. Interval arithmetic also conditions development of Artificial Intelligence and especially of automatic thinking, Computing with Words, grey systems, fuzzy arithmetic and probabilistic arithmetic. However, the mostly used conventional Moore-arithmetic has evident weak-points. These weak-points are well known, but nonetheless it is further on frequently used. The paper presents basic operations of RDM-arithmetic that does not possess faults of Moore-arithmetic. The RDM-arithmetic is based on multi-dimensional approach, the Moore-arithmetic on one-dimensional approach to interval calculations. The paper also presents a testing method, which allows for clear checking whether results of any interval arithmetic are correct or not. The paper contains many examples and illustrations for better understanding of the RDM-arithmetic. In the paper, because of volume limitations only operations of addition and subtraction are discussed. Operations of multiplication and division of intervals will be presented in next publication. Author of the RDM-arithmetic concept is Andrzej Piegat.

Keywords: interval arithmetic, RDM-interval arithmetic, multi-dimensional interval arithmetic, interval mathematics, interval analysis, granular computing, artificial intelligence

1. Introduction

Interval arithmetic comprises basic operations as addition, subtraction, multiplication and division of intervals. With its use one can e.g. add two quantities $a$ and $b$, values of which are not precisely but only approximatively known and the approximation has form of interval, e.g. $a \in [1, 3]$ and $b \in [3, 5]$. Interval arithmetic seems to be a less important area of mathematics and many students, engineers and scientists do not use it or even do not know about its existence. Meanwhile, the interval arithmetic has become a very important branch of mathematics in consequence of realization by many engineers and scientists of the fact that for achieving more credible problem solutions one should use any available information piece about a problem. Not only numerical and precise but also all approximate data pieces should be used. This aim was formulated e.g. for the famous and rapidly developing Grey Systems Theory [8] by its creator Professor Yulong Deng, a theory that undoubtedly can be called ”mathematics of the future”. Similar aims has Granular Computing [11]. Interval approximations probably are approximation forms the most frequently used in practice. Any technical measurement can
be formulated in the interval form, also evaluations (human measurements) made by problem experts. The interval range results from measurement error characteristic. In practice all, or almost all continuous variables as e.g. temperature are measured with an error. Thus, they cannot be precisely known. Only discrete variables as e.g. sum of money in our wallet can be measured precisely. In scientific investigations, in engineering, in economy, in medicine, etc, mathematical models contain variables and coefficients. At present, in problem solving, usually only precise knowledge of variable- and of parameter values is assumed. However, the so calculated results often considerably differ from real results. The reason of this state of matter is ignoring data uncertainty and introducing in mathematical models only variables, which are known "precisely" (though frequently the precision is an illusion only). If variables, the values of which are known approximately are not taken into account, then dimensionality of a model is reduced and this reduction can result in great quantitative and qualitative errors (e.g. the modeled system is nonlinear, its dimensionally reduced model is a linear one). Because in practice most variables and model parameters are known only approximately the interval arithmetic has application almost everywhere. Let us consider as example the car dynamics. On a car of mass $m[\text{kg}]$ acts a driving force $F[\text{N}]$. How large will be the car acceleration $a[\text{m/s}^2]$? The problem can be solved with use of Newton-formula $F = ma$ (1).

$$a = F/m$$ (1)

However, let us notice that in practice the car mass is not precisely known. The mass of author's empty car equals 1365 kg. When driving, there can be 1 to 5 people in the car (from 70 to 400 kg), in the trunk can be from 0 to 150 kg, in the fuel tank can be from 5 to 60 kg fuel. Thus, the real car-mass varies in interval $m \in [1440, 1975] = [m, \bar{m}]$. The force $F$ driving the car also is not precisely known because it depends on the present fuel quality (fuel quality varies), from air humidity, temperature and oxygen content. Thus, this force can be evaluated only approximately as $F \in [F, \bar{F}]$. The above shows that in practice we cannot base the acceleration calculations on formula (1) $a = F/m$ but they should be based on the interval formula (2).

$$[a] = \left[\frac{F, \bar{F}}{m, \bar{m}}\right]$$ (2)

Let us now consider another example taken from [13].

"There are 1000 chickens raised in a chicken farm and they are raised with two kinds of forages - soja and millet. It is known that each chicken eats 1.0 - 1.3 kg of forage every day and that for good weight gain it needs at least 0.21-0.23 kg of protein and 0.004-0.006 kg of calcium every day. Per kg soja contains 48-52% protein and 0.5-0.8% calcium and its price is 0.38-0.42 Yuan. Per kg, millet contains 8.5-11.5 protein and 0.3% calcium and its price is 0.20 Yuan. How should the forage be mixed in order to minimize expense on forage?"

Let us denote by $x_1$ the weight [kg] of soja that every day should be bought for the 1000 chicken and by $x_2$ the weight of millet. To determine the optimal values $x_{1opt}$ and $x_{2opt}$ the problem (3) should be solved.

Minimize the cost function:

$$Z[\text{Yuan}] = [0.38, 0.42] x_1 + 0.2 x_2$$ (3)
subject to:
\[ x_1 + x_2 = 1000[1.0, 1.3] \]
\[ [0.48, 0.52]x_1 + [0.085, 0.115]x_2 \geq 1000[0.21, 0.23] \]
\[ [0.005, 0.008]x_1 + 0.003x_2 \geq 1000[0.004, 0.006] \]
\[ x_1 \geq 0, x_2 \geq 0 \]

The above problem cannot be formulated and solved in terms of classical mathematics based on precise data knowledge. The problem can only be solved with use of Granular Computing [11] that among other things contains interval analysis. One can give a large number of examples illustrating the necessity of using interval mathematics instead of the classical mathematics of precise numbers. The interval mathematics is also necessary in various new science branches as e.g. artificial intelligence. An important area of artificial intelligence is fuzzy arithmetic [4, 7, 12] in the frame of which an interval-based calculation method called \( \alpha \)-cut method is used. Next example is probabilistic arithmetic [5, 6, 16] in which operations on distribution supports require application of interval arithmetic. The arithmetic also be used in the case of word-models in frame of Computing with Words [1, 17]. It is a very important branch of artificial intelligence that conditions creation of the automatic thinking similar to the human one. Interval arithmetic is necessary for almost all problems with uncertain, approximate information. However, the at present mostly used interval arithmetic type (examples can be books [3, 8, 10, 11, 13]) is further on Moore’s arithmetic [9, 10, 11], in spite of its known faults that are tried to be improved with different, sometimes very interesting but rather generally ineffective methods [3, 15, 14]. Further on there will be presented proposal of a new interval arithmetic, which is free from faults of Moore-arithmetic. This arithmetic is based on RDM-variables (Relative-Distance-Measure variables) and on multidimensional approach to interval operations. To give credence to this new arithmetic type a testing method is proposed that allows for checking calculation results delivered by any type of interval arithmetic, not only by the RDM-one.

2. Addition of intervals

Addition operation of two precise numbers \( a + b = x = ? \) can be called "forward calculation", because this operation can be interpreted as processing of input signals \( a \) and \( b \) realized by certain object, Fig. 1.

![Figure 1. Illustration of forward-calculations (a and b known, x unknown) and backward-calculations (a and c known, x unknown).](image)

Forward and backward calculations in case of precise numbers do not cause any difficulties. If e.g. two numbers should be added: \( a = 2, b = 3 \) then \( a + b = x = 5 \) (forward calculation). If \( a = 5 \) and \( c = 9 \) then \( x = c - a = 4 \) (backward calculation). However, difficulties appear when the values \( a, b, c \) taking part in the addition process are not precisely but only approximately known and this approximation is of interval character, e.g.
\[ [a] = [a, \bar{a}] = [3, 5], \text{ where } a \text{ and } \bar{a} \text{ appropriately mean the lower and upper limit of interval.} \]

For the addition operation Moore gave formula (4).

\[
[a, \bar{a}] + [b, \bar{b}] = [a + b, \bar{a} + \bar{b}] = [x, \bar{x}] \quad (4)
\]

Example of addition:

\[ [0, 2] + [1, 4] = [1, 6] \]

Results of addition according to Moore-formula (4) are intuitively fully understandable.

Let us apply the Moore-formula for a backward-calculation (5).

\[ [0, 2] + [x, \bar{x}] = [1, 6] \quad (5) \]

On the basis of formula (4) formula (6) can be written that allows for the solution calculation \([x, \bar{x}]\).

\[
\begin{align*}
0 + x &= 1, x = 1 \\
2 + \bar{x} &= 6, \bar{x} = 4 \\
[x] &= [x, \bar{x}] = [1, 4]
\end{align*} \quad (6)
\]

However, is the achieved result correct? Not quite! Let us notice, that a possible solution of equation (5) can be the number pair \(a = 0\) and \(x = 6\). But the solution \([x, \bar{x}] = [1, 4]\) does not contain the value \(x = 6\). It means that this solution, sometimes called in the literature [3] equation root is not correct. This example shows that the conventional Moore-arithmetic has limited possibilities an generally does not allow for backward calculations or, with other words, for equation solving that in practical applications occurs frequently. What is reason of this fault? To explain this question a new concept has to be introduced. It is the concept of RDM-variables (Relative-Distance-Measure variables). If the precise value of variable \(x\) is not known but we know the interval \([x, \bar{x}]\) which contains this value, then a new variable \(\alpha_x\) can be introduced that satisfies condition \(\alpha_x \in [0, 1]\) and the original interval can be expressed in form of (7).

\[ x = x + \alpha_x (\bar{x} - x), \alpha_x \in [0, 1] \quad (7) \]

If e.g. \(x \in [3, 5]\) then this information can be expressed as

\[ x = 3 + 2\alpha_x, \alpha_x \in [0, 1] \]

The interval notation (7) is illustrated by Fig. 2.

\[
\begin{align*}
\text{Figure 2. Illustration of meaning of the RDM-variable } \alpha_x &\text{ in case of a normal interval } [x, \bar{x}], x \leq \bar{x}. \\
\end{align*}
\]

Let us once more consider addition of two intervals (8).

\[
[a, \bar{a}] + [b, \bar{b}] = [x, \bar{x}] = ? \quad (8)
\]

Using RDM variables equation (8) can be transformed in (9).

\[
\frac{a + \alpha_a (\bar{a} - a)}{\alpha_a \in [0, 1]}, \frac{b + \alpha_b (\bar{b} - b)}{\alpha_b \in [0, 1]} = x \quad (9)
\]
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Table 1. Addition results $x$ of variables $a$ and $b$ for various values of the RDM-variables $\alpha_a$ and $\alpha_b$ in general and in number-form for $[a, \alpha_a] = [0, 2]$ and $[b, \beta_b] = [1, 4]$.

<table>
<thead>
<tr>
<th>$\alpha_a$</th>
<th>$\alpha_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

$x = (a + b)$ | $x = (\alpha + \beta)$ | $x = (\pi + b)$ | $x = (\pi + \beta)$

| $x$ | 1 | 4 | 3 | 6 |

Depending on values of variables $\alpha_a$ and $\alpha_b$ the resulting variable $x$ assumes various values shown in Table 1.

As results from Table 1 the minimal result value equals $x = \alpha + \beta = 1$ and the maximal value $x = \alpha + \beta = 6$. This result is compatible with the Moore-arithmetic one $[1, 4]$. The analyzed problem is illustrated in Fig. 3.

Fig. 3 shows that rectangular knowledge-granule (input-granule) $[a, \alpha_a] \times [b, \beta_b] = [0, 2] \times [1, 4]$, as Cartesian product of intervals $[a] \times [b]$ cuts on the addition surface $a + b = x$ a 3D-granule of solution (output-granule). Though the 3D-picture of the output-granule explains well the intervals’ addition, this operation almost equally well can be presented in 2D-space as in Fig. 4.
The knowledge granule \([a] \times [b]\) shown in Fig. 4 is cut by contour lines of constant values of sum \(a + b = x = \text{const}\), e.g. \(a + b = 3, a + b = 4\), etc. One can easily notice that these lines are of different length. The shortest lines are \(a + b = 1\) and \(a + b = 6\) (1-element sets of solutions). The longest lines are lines corresponding to values \(x \in [3, 4]\). Length of a contour line represents measure of a solutions’ set, e.g. length of the line \(a + b = 3\) represents measure of all tuples \((a, b)\) satisfying the condition \(a + b = 3\). The contour-lines’ length can be interpreted as non-normalized, a priori probability density of the event \(a + b = x\). Subject to assumption of the uniform distribution of probability density for variables \(a\) and \(b\), on the basis of Fig. 4 the distribution of a priori probability density for the addition result \(a + b = x\) shown in Fig. 5 can be achieved.

Figure 5. Distribution of a priori probability density (Fig. 5c) of the addition result of two intervals with use of the RDM-method achieved at assumption of uniform distributions for components \(a\) and \(b\) (Fig. 5a and Fig. 5b).

It should be taken into account that the trapezoidal distribution from Fig. 5c was achieved at assumption of uniform distributions for components \(a\) and \(b\) in situation when experimental distributions are not known. Similarly, if for a coin experimental probabilities of head and tail are not known equal a priori probabilities are assumed 0.5 for head and 0.5 for tail. In the case, when experimental distributions are known then they should be used for determining the distribution of the sum \(x = a + b\). Let us consider now backward calculations with use of RDM-variables. As an example of forward calculations the addition problem as below was solved.

\[
[a, \bar{a}] + [b, \bar{b}] = [0, 2] + [1, 4] = [x, \bar{x}] = [1, 6]
\]

In frame of backward calculations problem (10) will be considered.

\[
[a, \bar{a}] + [x, \bar{x}] = [c, \bar{c}]
\]

\[
[0, 2] + [x, \bar{x}] = [1, 6]
\]

(10)

This problem is illustrated by Fig. 6.

As was earlier shown, application of Moore-arithmetic gives the incorrect solution \([x, \bar{x}] = [1, 4]\). Let us now solve this problem with use of the RDM-arithmetic. To this aim RDM-variables are introduced \(\alpha_a \in [0, 1]\) and \(\alpha_c \in [0, 1]\) and equation (10) is appropriately transformed in (11).

\[
\begin{align*}
\alpha_a (\bar{a} - a) + x &= \bar{c} + \alpha_c (\bar{c} - c) \\
0 + 2\alpha_a + x &= 1 + 5\alpha_c \\
\alpha_a &\in [0, 1], \alpha_c \in [0, 1]
\end{align*}
\]
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Figure 6. Illustration of interval backward-calculations where intervals $[a, \overline{a}]$ and $[c, \overline{c}]$ are known and interval $[x, \overline{x}]$ is not known.

The above equations are usual mathematical equations and can be solved with use of classical mathematics. The solution is given by equation (12).

\[
x = c - a + 5c (c - a) - \alpha_a (\overline{a} - a) \\
x = 1 + 5\alpha_c - 2\alpha_a \\
\alpha_a \in [0, 1], \alpha_c \in [0, 1]
\]

Table 2 shows values of the result $x$ for various values of RDM-variables.

Table 2. General and numerical values of the result $x$ of equation $[a, \overline{a}] + [x, \overline{x}] = [c, \overline{c}]$.

<table>
<thead>
<tr>
<th>$\alpha_a$</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_c$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$x$</td>
<td>(c - a)</td>
<td>(c - a)</td>
<td>(c - a)</td>
<td>(c - a)</td>
</tr>
</tbody>
</table>

Fig. 7 shows the input granule $[a, \overline{a}] \times [c, \overline{c}] = [0, 2] \times [1, 6]$ in 3D-space $A \times C$, and Fig. 8 the output granule in 2D-space: $A \times X$.

Figure 7. Input-(knowledge) granule $[a, \overline{a}] \times [c, \overline{c}] = [0, 2] \times [1, 6]$ in 3D-space $A \times C$.

After projecting the input granule on the functional addition surface a 3D-solution granule is achieved which is next projected on 2D-surface $A \times X$ in Fig. 8.

It should be noticed that after solving equation $[0, 2] + [x, \overline{x}] = [1, 6]$ with Moore-arithmetic the 1-dimensional solution $([x, \overline{x}] = [1, 4]$) is obtained. This solution is not complete because it does not contain e.g. the test-point $TP(0.5, 5)$ though this point satisfies the considered equation $[0, 2] + [x, \overline{x}] = [1, 6]$. The sum $0.5 + 5.0 = 5.5$ is number lying in the interval $[1, 6]$. Instead, solution (12) delivered by the RDM-arithmetic is 2-dimensional because 2 variables $\alpha_a$ and $\alpha_c$ occur in it.

\[
x = 1 - 2\alpha_a + 5\alpha_c
\]
Let us see in Fig. 8 that the solution granule cannot in any way be presented (described) as a 1-dimensional granule \([x; x]\). Thus, the conventional interval arithmetic is not able to solve backward-calculations (equation solving). In many cases it delivers false or even paradoxical solutions and persons achieving these solutions are not conscious of this fact and use them in real problems. One of false conclusions suggested by Moore-arithmetic is so-called the principle of increasing entropy [3].

3. Interval Moore-arithmetic and the principle of increasing entropy

Let us once more consider the addition-operation of intervals according to Moore-arithmetic (13).

\[
[a, \alpha] + [b, \beta] = [a + b, \alpha + \beta]
\]

\[
[0, 2] + [1, 4] = [1, 6]
\]

This operation is illustrated in Fig. 9.

Let us notice that width of the resulting interval \([x, \pi]\) equal in the considered case to 5 is equal to the sum of widths of the added intervals \([a]\) and \([b]\) i.e. (2+3). The width growing of resulting intervals intuitively is fully understandable and in the subject literature it is called the principle of increasing entropy. Further on a quotation from [3] is given.

"...the rules of interval mathematics are constructed in such a way that any arithmetical operation on intervals results in an interval as well. These rules conform to the well known common view-point that any arithmetical operation with uncertainties should increase the total uncertainty (and entropy) of the system". Now, let us consider the question, whether such situation is possible, that after adding two intervals a resulting interval will be achieved with width that is smaller than widths of two components , i.e. whether a result is possible that contradicts the increasing entropy (uncertainty) principle? E.g. is the addition result presented by (14) possible?

\[
[a] + [x] = [c] = [1, 9] + [x; \pi] = [11, 12]
\]
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Width of interval \([a]\) in (14) equals 8, width of interval \([x]\) has to be positive, and width of the resulting interval \([c]\) equals 1. Solving equation (14) with use of Moore-arithmetic the result \(\bar{x} = 10\) and \(\underline{x} = 3\) is achieved.

\[
[\bar{x}, \underline{x}] = [10, 3]
\]

This solution is absurd because the lower interval-limit \(\underline{x} = 10\) is greater than the upper limit \(\bar{x} = 3\). Let us now consider Example 1.

**Example 1.** From field 1 crop of wheat was brought to a warehouse. The crop \(a\) weighed on the field with a simple, inaccurate balance of the maximal error \(\pm 100 kg\) belongs to interval \(a \in [4900, 5100]\). From field 2 also a crop of wheat was brought, but of an unknown weight \(x\). Both crops \(a\) and \(x\) were weighed together in the warehouse with a balance of the maximal error \(\pm 50 kg\). The weighing delivered information that the total weight \(c = a + x\) belongs to interval \(c \in [8950, 9050]\). The weight \(x\) of the crop from field 2 should be determined.

Mathematical formulation of Example 1 is given by (15).

\[
[4900, 5100] + [\bar{x}, \underline{x}] = [8950, 9050]
\] (15)

Uncertainty of the left-hand side of equation (15) equals at least 200 and is higher than uncertainty of the right-hand side, which equals 100. Equation (15), if considered purely theoretically seems absurd. However, situation described in Example 1 is fully real and possible. Thus, equation (14) describes situation that is ”contradictory” with the increasing entropy (uncertainty) principle. Let us try to solve this equation with Moore-arithmetic. The solution is given by (16).

\[
[\bar{x}, \underline{x}] = [4050, 3950]
\] (16)

Solution (16) is absurd because the lower limit \(\underline{x}\) exceeds the upper limit \(\bar{x}\). Let us now solve Example 1 with use of RDM-variables. Interval \([a]\) is transformed in form (17) and interval \([c]\) in form (18).

\[
[a, \bar{a}] = [4900, 5100] = 4900 + 200\alpha_a, \alpha_a \in [0, 1]
\] (17)

\[
[c, \bar{c}] = [8950, 9050] = 8950 + 100\alpha_c, \alpha_c \in [0, 1]
\] (18)

Equation (15) takes form of (19).

\[
x = 4050 - 200\alpha_a + 100\alpha_c, \alpha_a \in [0, 1], \alpha_c \in [0, 1]
\] (19)

Solution (19) achieved with the RDM-method is 2-dimensional and solution (16) achieved with Moore-method is 1-dimensional. Table 3 shows characteristic values of variable \(x\) for various border-values of RDM-variables \(\alpha_a\) and \(\alpha_c\).

<table>
<thead>
<tr>
<th>(\alpha_a)</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha_c)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(x)</td>
<td>(c - a)</td>
<td>(\bar{a} - a)</td>
<td>(c - \bar{a})</td>
<td>(\bar{c} - \bar{a})</td>
</tr>
<tr>
<td>(x)</td>
<td>4050</td>
<td>4150</td>
<td>3850</td>
<td>3950</td>
</tr>
</tbody>
</table>

Fig. 10 illustrates the considered problem in 3D-space. Fig. 10 shows that to find solution of the considered interval-equation, in the first step, the knowledge granule has to be projected on the functional surface of addition \(a + x = c\).
Figure 10. Visualization of solving operation of the interval equation \([a] + [x] = [c]\), where intervals \([a]\) and \([c]\) are known and interval \([x]\) is unknown.

Figure 11. Illustration of solution of the interval-equation \([a] + [x] = [c]\) = \([4900; 5100]\) + \([x, \bar{x}] = \[8950; 9050]\).

This operation yields 3D-solution of the equation. In the next step the 3D-granule of solution should be projected on space \(A \times X\), which delivers the 2D-solution. This solution-granule is shown in Fig. 11.

As Fig. 11 shows, solution of the interval equation \([a] + [x] = [c]\), in the general case, is not 1-dimensional and cannot be written in form \([x, \bar{x}]\) as suggested by conventional Moore-arithmetic. This solution exists only in 2D-space and can only be described with use of two RDM-variables \((20)\) or with use of variables \(a\) and \(x\) \((21)\).

\[
x = 4050 - 200\alpha_a + 100\alpha_c, \alpha_a \in [0, 1], \alpha_c \in [0, 1]
\]

\[
a + x \geq 8950 \\
a + x \leq 9050, a \in [4900, 5100]
\]

It means that the way of writing the interval equation \([a] + [x] = [c]\), for years suggested by Moore-arithmetic, is incorrect. The correct notation is given by \((22)\),

\[
[a] + [(a, x)] = [c]
\]
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incorrect notation form of interval equation suggested by Moore-arithmetic has moved many scientist in a wrong direction of searching for 1-dimensional solution \([x, x]\). In Example 1 there was shown that contradictory to the principle of increasing entropy solving the equation (15) is possible.

\[
[4900, 5100] + [x, x] = [8950, 9050]
\]

In this equation uncertainty of the left-hand side (200) is 2 times greater than the result uncertainty (100) on the right-hand side. However, it is also possible to solve with the RDM-arithmetic such equation where uncertainty of the addition result equals zero. Let us consider Example 2.

Example 2. From field 1 crop of wheat was brought to a warehouse. The crop \(a\) weighed on the field with a balance of the maximal error \(\pm 100\, \text{kg}\) belongs to interval \(a \in [4900, 5100]\). From field 2 also a wheat crop was brought, but of an unknown weight \(x\). Both crops \(a\) and \(x\) had been weighed together in the warehouse with an ideal balance of zero error. The weighing result was \(a + x = c = 9000\, \text{kg}\). The weight \(x\) of the crop from field 2 should be determined.

In Example 2 an interval equation (23) should be solved.

\[
[a] + [x] = [c] \\
[4900, 5100] + [x, x] = [9000, 9000]
\]

Solution of equation (23) consists of a set of tuples \((a, x)\) that satisfy dependence (24) written with use of RDM-variable \(\alpha_a\) or dependence (25) written with use of variables \(a\) and \(x\).

\[
x = 4100 - 200\alpha_a, \quad a = 4900 + 200\alpha_a, \quad \alpha_a \in [0, 1]
\]

\[
x = 9000 - a, \quad a \in [4900, 5100]
\]

Correct notation of equation (23) is given in form of equation (26).

\[
[4900, 5100] + [(a, x)] = [9000, 9000]
\]

Solution of Example 2, which fully contradicts the principle of increasing entropy is shown in 2D-space \(A \times X\) in Fig. 12.

![Figure 12](image-url)

Figure 12. Visualization of solution of interval equation (23) with zero-uncertainty on the right-hand side \([4900, 5100] + [(a, x)] = [9000, 9000]\).

Fig. 12 explains that variables \(a\) and \(x\) were completely correlated one with another. Only in this case the result \([c, c] = [9000, 9000]\) could possess zero-uncertainty. With other
words, variables $a$ and $x$ were ideally fitted one to another. On the other hand, solution of Example 1 shown in Fig. 11 explains that if uncertainty of the right-hand side of interval equation $([4900,5100] + [(a, x)] = [8950,9050])$ is smaller than uncertainty of the left-hand side then variables $a$ and $x$ are partly (not fully) correlated (are partly fitted one to another). Moore-arithmetic is able to correctly solve only such equations $[a]+[x]=[c]$ for which 1-dimensional solution $[x, \bar{x}]$ exists, for case, when both variables $a$ and $x$ are completely not correlated (zero-correlation).

4. Operation of interval subtraction

According to Moor’s interval arithmetic subtraction of intervals $[a] - [b] = [x]$ should be realized with use of formula (27).

$$[a, \bar{a}] - [b, \bar{b}] = [a - \bar{b}, \bar{a} - b] = [x, \bar{x}]$$ (27)

Fig. 13 shows a fragment of the subtraction functional-surface $a - b = x$ in 3D-space.

![Subtraction Surface](image)

**Figure 13.** Fragment of the functional subtraction-surface $a - b = x$ with contour-lines of constant difference values $a - b = x = \text{const}$.

The functional subtraction-surface can be projected from 3D-space into 2D-space ($A \times B$), what is shown in Fig. 14.

![Projection](image)

**Figure 14.** Projection of subtraction-surface $a - b = x$ from 3D-space $A \times B \times X$ (Fig.13) on 2D-space $A \times B$.

Let us consider now Example 3 of forward-calculations.
**Example 3.** To a warehouse a wheat transport was brought of weight $a \in [2900, 3100]\, kg$ that had been weighed on the field with a simple field balance. From the wheat transport a part $b$ was sold to a customer that was weighed with a balance of smaller error than the field balance. The weight $b$ belongs to interval $b \in [2200, 2300]\, kg$. How large is the wheat weight left in the warehouse?

To find solution of the problem equation (28) should be solved.

$$[a, \bar{a}] - [b, \bar{b}] = [a - \bar{b}, \bar{a} - b] = [x, \bar{x}]$$

(28)

With use of Moore-arithmetic solution (29) is obtained.

$$[2900, 3100] - [2200, 2300] = [600, 900]$$

(29)

With use of RDM-variables $\alpha_a$ and $\alpha_b$ equation (28) can be transformed in equation (30).

$$\begin{align*}
(2900 + 200\alpha_a) - (2200 + 100\alpha_b) &= 700 + 200\alpha_a - 100\alpha_b = x \\
\alpha_a &\in [0, 1], \alpha_b \in [0, 1]
\end{align*}$$

(30)

Value $a - b = x$ will be maximal for $\alpha_a = 1$ and $\alpha_b = 0$ ($x = 900$). The minimal value $x$ is achieved for $\alpha_a = 0$ and $\alpha_b = 1$ ($x = 600$). Thus, solution $[x, \bar{x}] = [600, 900]$ delivered by Moore-arithmetic is the same as the one delivered by the RDM-arithmetic. Operation of interval-subtraction is illustrated in Fig. 15.

![Figure 15](image-url)  
*Figure 15. Illustration of solving Example 3 (forward-calculation) $[a] - [b] = [x] = [2900, 3100] - [2200, 2300]$: the solution is interval $[x] = [600, 900]$*

Lengths of particular contour lines in Fig. 15 inside of the input granule in Fig. 15 are proportional to a priori density of probability of resulting variable $x$. Therefore they generate distribution shown in Fig. 16.

As Fig. 16 shows, the RDM-arithmetic enables in this case achieving not only the correct range $[x, \bar{x}]$ of intervals’ subtraction $[a] - [b]$ but also a priori distribution of probability density $pd(x)$ owing to the fact that it realizes multi-dimensional approach to interval operations. Let us now consider the problem of backward calculations $[a] - [x] = [c]$.

**Example 4.** To a warehouse a wheat transport $a$ was brought from a field where it had been weighed with a simple field balance with the maximal error $\pm 100\, kg$. It results from the weighing that $a \in [2900, 3100]\, kg$. The following night a part $x[kg]$ of the wheat was stolen from the warehouse. The remainder $c$ was weighed with a balance of the maximal error $\pm 50[kg]$. It results from the weighing that $c \in [2200, 2300]\, kg$. How much wheat was stolen?
To answer the question with use of Moore-arithmetic equation (31) is to be solved.

\[
[2900, 3100] - [x, \overline{x}] = [2200, 2300]
\]  

(31)

Solution of equation (31) is given by (32).

\[
\begin{align*}
2900 - \overline{x} &= 2200, \overline{x} = 700 \\
3100 - x &= 2300, x = 800 \\
[x, \overline{x}] &= [800, 700]
\end{align*}
\]  

(32)

Solution (32) is absurd because \(x > \overline{x}\) that can be interpreted as negative width \((\overline{x} - x = 700 - 800 = -100)\) of the interval or negative entropy [3]. Now, let us solve Example 4 with use of the RDM-arithmetic. Interval \([a]\) can be written as

\[a = 2900 + 200\alpha_a, \alpha_a \in [0, 1].\]

Interval \([c]\) can be written as

\[c = 2200 + 100\alpha_c, \alpha_c \in [0, 1].\]

And equation (31) can be written in form of (33)

\[
(2900 + 200\alpha_a) - x = 2200 + 100\alpha_c \\
x = 700 + 200\alpha_a - 100\alpha_c, \alpha_a, \alpha_c \in [0, 1], \alpha_c \in [0, 1]
\]  

(33)

Equation (33) is the mathematical solution of the problem. Table 4 shows values of variable \(x\) for various border values of \(\alpha_a\) and \(\alpha_c\).

Table 4. Values of variable \(x\) for various border-values of RDM-variables \(\alpha_a\) and \(\alpha_c\), and corresponding values of variables \(a\) and \(c\).

| \(\alpha_a\) | 0 | 0 | 1 | 1 |
| \(\alpha_c\) | 1 | 1 | 0 | 0 |
| \(x\) | 600 | 700 | 800 | 900 |
| \(a\) | 2900 | 2900 | 3100 | 3100 |
| \(c\) | 2300 | 2200 | 2300 | 2200 |
| \(x\) | 600 | 700 | 800 | 900 |

Fig. 17 presents solution granule of equation \([2900, 3100] - [(a, x)] = [2200, 2300]\) from Example 4.
Is the conventional interval arithmetic correct?

One can easily check whether solution presented in Fig. 17 corresponding to equation (33) is correct. E.g. the point \((a = 7100, x = 600)\) shouldn’t satisfy equation (31) and point \((a = 2900, x = 600)\) should. Moore-arithmetic is not able to define zero-point (neutral point). In the traditional non-interval arithmetic zero-point can be defined as difference of two identical numbers.

\[ a - a = 0 \]

One can similarly try to define the interval zero-element as difference of two identical intervals \([a, \bar{a}]\) and \([\bar{a}, \alpha]\), formula (34).

\[ [a, \bar{a}] - [\bar{a}, \alpha] = [(a - \bar{a}) , (\bar{a} - a)] = [-(\bar{a} - a) , (\bar{a} - \alpha)] \tag{34} \]

Result (34) is an interval of width \(2(\bar{a} - a)\). Thus, it is not the neutral zero-element. However, with use of the RDM-arithmetic the neutral element can be defined. Let us consider Example 5.

Example 5. To a warehouse a wheat transport \(a[kg]\), \(a \in [2900, 3100]kg\), was brought from a field. The transport was as a whole bought by a customer. What amount \(x[kg]\) was left in the warehouse?

If the conventional Moore-arithmetic is used for the problem solution then equation (35) has to be used.

\[ [a, \bar{a}] - [a, \bar{a}] = [x, \bar{x}] \tag{35} \]

Conventional solution of the problem is given by (36).

\[ [a, \bar{a}] - [a, \bar{a}] = [- (\bar{a} - a) , (\bar{a} - \alpha)] = [x, \bar{x}] \]

\[ [2900, 3100] - [2900, 3100] = [-200, 200] = [x, \bar{x}] \tag{36} \]

It means that in the warehouse certain amount of wheat \(x \in [-200, 200]\) was left that can be different from zero. This conclusion is rather illogical. To solve the problem with the RDM-arithmetic variable \(\alpha_a \in [0, 1]\) is introduced. The weight \(a[kg]\) of wheat, which is known only approximately can be expressed by dependence (37).

\[ a = a + \alpha_a (\bar{a} - a) = 2900 + 200\alpha_a, \alpha_a \in [0, 1] \tag{37} \]

Though the weight \(a\) of the wheat isn’t known precisely, this weight possess only one concrete value lying in interval \(a \in [2900, 3100]\). This value can e.g. be equal to \(a =
2951.67132\ldots kg. Independently of what amount \(a[kg]\) had been brought to the warehouse from the field, the same amount \(a[kg]\) was taken by the customer. Thus, zero \(kg\) of wheat was left in the warehouse and, correctly, the problem should be expressed by dependence (38).

\[
[a] - [a] = [a + \alpha_a (\bar{a} - a)] - [a + \alpha_a (\bar{a} - a)] = 0, \alpha_a \in [0, 1]
\] (38)

Summarizing: there are problems in which difference of two identical intervals is precisely equal to zero and there are problems in which difference of two identical intervals is not equal to zero and is represented by an interval \([\bar{a}, \bar{a}]\). This fact means a great difference between the classical singleton-mathematics and the interval mathematics. Calculation results of the singleton-mathematics are general and problem-independent. However, in case of interval mathematics calculation results are problem-dependent. Thus, interval mathematics is much more difficult and complicated than singleton-mathematics. In Example 6 a problem will be shown, where difference of two identical intervals is not equal to zero.

**Example 6.** Boxer \(A\) and boxer \(B\) before fight were weighed with a balance of maximal error \(\pm 0.5kg\). In case of both boxers the balance shown the same value of \(85 kg\). It means that the real weight \(a\) of boxer \(A\) lies in interval \(a \in [84.5, 85.5]\) and the real weight \(b\) of boxer \(B\) also lies in the same interval \(b \in [84.5, 85.5]\). How large is the weight difference between boxer \(A\) and \(B\)?

To determine this difference the difference of two intervals should be found as below.

\[
[a, \bar{a}] - [b, \bar{b}] = [84.5, 85.5] - [84.5, 85.5]
\]

Weight of each boxer has a concrete value. Weight of boxer \(A\) can be e.g. \(a = 84.791\ldots kg\) and of boxer \(B\) can be e.g. \(b = 85.123\ldots kg\). Probability that both weights are ideally equal is infinitely small. Weight of boxer \(A\) can be written as below.

\[
a = a + \alpha_a (\bar{a} - a) = 84.5 + \alpha_a, \alpha_a \in [0, 1]
\]

Weight of boxer \(B\) can be written as

\[
b = b + \alpha_b (\bar{b} - b) = 84.5 + \alpha_b, \alpha_b \in [0, 1]
\]

Thus, the weight difference is determined by formula (39).

\[
[a] - [b] = 84.5 + \alpha_a - 84.5 - \alpha_b, \alpha_a \in [0, 1], \alpha_b \in [0, 1]
\]

\[
[a] - [b] = [\alpha_a - \alpha_b]
\]

\[
[a] - [b] = [-1, 1]
\] (39)

It means that the maximal weight-difference of the boxers can reach even \(1kg\), in spite of the fact that the balance shown the same weight of \(85kg\) for both boxers. Summarizing Example 5 and Example 6 one can say that in case of interval arithmetic it is not possible to give one general formula as in the traditional singleton-arithmetic (e.g. \(a - a = 0, 5 - 5 = 0\)) for subtraction of two identical intervals (e.g. \([a] - [a] = [2, 3] - [2, 3]\)) that would be correct in all cases. In each real problem one should consider whether the real (not precisely but only approximately known) subtracted values \(a - b\), represented by intervals \([a, \bar{a}]\) and \([b, \bar{b}]\) are identical or different. Thus, interval arithmetic is much more complicated than the arithmetic of precise, non-interval numbers.
5. Conclusions

The paper presented a new (according to authors’ knowledge) multi-dimensional interval arithmetic based on RDM-variables for case of addition and subtraction (because of the paper-volume limitation). Operations of multiplication and division will be described in next paper of authors. There was shown, that uncertain, approximate parameters of a system model increase its dimensionality what next increases calculation difficulties. However, the RDM-arithmetic allows for correct solutions of such problems, which cannot be solved by Moore’s interval arithmetic. It was shown in the paper that the principle of increasing entropy (uncertainty) of interval calculations is generally not true. It also was shown that interval arithmetic mainly realizes the task of function-extrema determining in a constrained granule-space. The RDM-arithmetic operations presented in the paper are only beginning of large investigations that will lead to a new multidimensional interval-mathematics which enables solving many complicated problems containing uncertainty and approximate data, especially problems of artificial intelligence, fuzzy mathematics, probabilistic mathematics, Computing with Words etc. Especially the $\alpha$-cut method used in fuzzy arithmetic will require a revision to correctly solve equations.

References
